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LIMIT THEOREMS FOR PERIODIC QUEUES

by

J. MICHAEL HARRISON and AUSTIN J. LEMOINE

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1. Introduction and Summary

Let $A = \{A(t), t \geq 0\}$ be a non-stationary Poisson process with bounded measurable arrival rate function $\lambda(\cdot)$ satisfying $\lambda(n + t) = \lambda(t)$ for all t in $[0, 1)$ and $n \in I = \{1, 2, \ldots\}$. We call the unit of time a day, and we refer to a point t in $[0, 1)$ generically as a time of day. Defining

$$\lambda = \int_0^1 \lambda(t)\, dt \quad ,$$

we say that A is a periodic Poisson process with a period of one day and an average arrival rate of λ. We denote by $T_1, T_2, \ldots$ the jump points of the right-continuous process A.

Let $S_1, S_2, \ldots$ be independent and identically distributed (i.i.d) positive random variables, independent of A, with distribution function (d.f) F having $F(0) = 0$. We define $\rho = \lambda E\{S_1\}$ and assume throughout that $\rho < 1$. For $t > 0$ let $X(t) = S_1 + \ldots + S_{A(t)}$, with the convention that an empty sum is zero, and then let $Y(t) = X(t) - t$ and

$$Z(t) = Y(t) - \inf\{Y(s-) : 0 \leq s \leq t\} \quad .$$

We interpret $T_1, T_2, \ldots$ as the customer arrival times at an ordinary single server queueing system and $S_1, S_2, \ldots$ as the customer service times. Then $Z = \{Z(t), t \geq 0\}$ represents the server load process (or virtual waiting time process) for this system, assuming that the server is initially idle; cf. Beneš (1958). Alternatively, one may interpret Z as the contents process for a dam with input process $X = \{X(t), t \geq 0\}$ and (constant) unit release rate, assuming the dam is initially empty. If the queueing context, the actual waiting time of customer n is given by $W_n = Z(T_n-) = Z(T_n) - S_n$. It is the purpose of

this paper to study the asymptotic behavior of the processes Z and $W = \{W_n, n \in I\}$.

Queues with non-homogeneous Poisson arrivals have been studied earlier by Takács(1955), Reich (1958, 1959) and Hasofer (1964). Hasofer (1964) considered the case of periodic input and, imposing some further conditions on the arrival rate function and service time distribution, showed that the probability of server idleness is an asymptotically periodic function of time.

In Section 2 we focus attention on the integer time points (i.e., where days begin) at which the system is empty, showing that these times constitute a sequence of regeneration points for the continuous parameter process Z . A similar sequence of regeneration points is identified for the discrete parameter process W . In Section 3, these results are combined with discrete renewal theory to prove the following weak limit theorems. For each time of day $t \in [0, 1)$ there exists a proper distribution H_t such that $P\{Z(n + t) \leq x\} \to H_t(x)$ as $t \to \infty$ for all $x \geq 0$. Furthermore, there exists a proper distribution G such that $P\{W_n \leq x\} \to G(x)$ as $n \to \infty$ for all $x \geq 0$. We also prove strong limit theorems for Z and W . Specifically, for all $x \geq 0$,

$$(1) \qquad \frac{1}{t}\int_0^t 1_{\{Z(s) \leq x\}}\, ds \longrightarrow H(x) \quad \text{as} \quad t \to \infty$$

with probability one (w.p.1) and

$$(2) \qquad \frac{1}{n}\sum_{k=1}^{n} 1_{\{W_k \leq x\}} \longrightarrow G(x) \quad \text{as} \quad n \to \infty$$

w.p.1, where

$$H(x) = \int_0^1 H_s(x)\, ds \quad , \quad x \geq 0 \quad .$$

As a by-product of (2) we show that

$$G(x) = \frac{1}{\lambda}\int_0^1 \lambda(s)\, H_s(x)\, ds \quad , \quad x \geq 0 \quad .$$

Finally, in Section 4 we show that the limit distributions G and H are related by

$$(3) \qquad H(x) = 1 - \rho + \rho \cdot (G * \hat{F})(x) \quad , \quad x \geq 0 \quad ,$$

where $*$ denotes convolution and the d.f. $\hat{F}$ is defined by

$$(4) \qquad E\{S_1\} \cdot \hat{F}(x) = \int_0^x [1 - F(y)]\, dy \quad , \quad x \geq 0 \quad .$$

In particular, this shows that $H(0) = 1 - \rho$, which is more or less obvious from physical considerations. The relationship (3) between the asymptotic virtual and actual waiting time distributions is exactly the same as that found by Takács (1963) for the GI/G/1 queue. Our line of proof follows Lemoine (1974) and shows that Takács' result actually holds for much more general systems if one interprets the distributions H and G as sample path limits as in (1) and (2).

The results presented here are weak in the sense that we neither compute nor characterize the distributions G and H , except to show that each can be obtained from the other via (3). By demonstrating that interesting asymptotic distributions do exist for periodic systems, however, we hope to suggest some potentially tractable problems in the theory of queues with non-stationary Poisson input and hence to rekindle interest in this important class of models.

2. The Regenerative Structure

We formally assume a probability space $(\Omega, \mathcal{F}, P)$ on which is defined a right-continuous compound Poisson process $X^* = \{X^*(t), t \geq 0\}$ with unit jump

rate and jump size distribution F. Let

$$\Lambda(t) = \int_0^t \lambda(u)\, du \quad , \quad t \geq 0 \quad .$$

Then, $\Lambda(\cdot)$ is non-decreasing and continuous with $\Lambda(0) = 0$ and $\Lambda(n + t) = n\lambda + \lambda(t)$ for $n \in I$ and $t \in [0, 1)$. Let $X(t) = X^*(\Lambda(t))$ for $t \geq 0$, and let $A(t)$ be the number of jumps of the process $X = \{X(u), u \geq 0\}$ during $[0, t]$. Thus, A and X are described in Section 1. Also, let

$$\mathscr{F}_t^* = \mathscr{F}(X^*(u) \ , \ 0 \leq u \leq t) \quad , \quad t \geq 0 \quad ,$$

$$\mathscr{F}_\infty^* = \bigvee_{0 \leq t < \infty} \mathscr{F}_t^* \quad ,$$

and

$$\mathscr{F} = \mathscr{F}_{\Lambda(t)} \quad , \quad 0 \leq t \leq \infty \quad .$$

We say that a random variable $T : \Omega \to [0, \infty]$ is optional if $\{T \leq t\} \in \mathscr{F}_t$ for all $0 \leq t < \infty$. And, for T optional we define

$$\mathscr{F}_T = \{E : E \cap \{T \leq t\} \in \mathscr{F}_t \text{ for all } 0 \leq t < \infty\} \quad .$$

The process X^* is strong Markov (cf. Blumenthal and Getoor (1968), pp. 37-44), from which we easily obtain the following.

<u>Proposition 1</u> . If T is optional and finite w.p.1, then

$$P\left\{X(T+t) - X(T) = 0 \middle| \mathscr{F}_T\right\} = \exp\left\{-[\Lambda(T+t) - \Lambda(T)\right\}$$

w.p.1 for $t \geq 0$.

Proposition 2 . If N is integer-valued, optional, and finite w.p.1, then $\{X(N+t) - X(N),\ t \geq 0\}$ is independent of $\mathscr{F}_N$ and distributed as $\{X(t),\ t \geq 0\}$.

We define $Y(t)$, $Z(t)$, T_n and W_n in terms of X as in Section 1.. Let $\alpha = \inf\{n \in I : Z(n) = 0\}$, with $\alpha = \infty$ if the indicated set is empty. Since $Y(t)$ and $Z(t)$ are $\mathscr{F}_t$-measurable for $t \geq 0$, it follows that α is an integer-valued optional time. Recall that we assume $\rho = \lambda E\{S_1\} < 1$, and observe that $E\{X(1)\} = \rho$.

Proposition 3 . $E\{\alpha\} < \infty$.

Proof: Since $Y(t) = X^*(\Lambda(t)) - t$, we have $Y(t)/t \to \rho - 1$ as $t \to \infty$ (w.p.1), and hence $Y(t) \to -\infty$ as $t \to \infty$ (w.p.1). From the definition of Z it then follows that $\sup\{t : Z(t) = 0\} = \infty$ w.p.1. Moreover, it is clear that $\sup\{t : Z(t) > 0\} = \infty$ w.p.1. Let $\xi_1 = 0$ and recursively define

$$\nu_n = \inf\{t > \xi_n : Z(t) > 0\}$$

and

$$\xi_{n+1} = \inf\{t > \nu_n : Z(t) = 0\}$$

for $n \in I$. In queueing terminology, ξ_n is the time at which the nth idle period begins, and ν_n is the time at which the nth busy period begins, $n \in I$. From the remarks above, the variables ξ_n and ν_n are finite w.p.1 for $n \in I$, and each is clearly optional. Let $\gamma_n = \inf\{k \in I : k \geq \xi_n\}$ for $n \in I$, so that γ_n is the first integer time following the onset of the nth

idle period. Then $\alpha = \gamma_N$ where $N = \inf\{n \in I : \gamma_n \leq \nu_n\}$. Now,

$$P\{N \geq n+1\} = P\{N \geq n, \nu_n < \gamma_n\}$$

$$= P\{\alpha \geq \xi_n, X(\gamma_n) - X(\xi_n) > 0\}$$

$$= \int_{\{\alpha \geq \xi_n\}} P\{X(\gamma_n) - X(\xi_n) > 0 \mid \mathcal{G}_n\}\, dP \quad .$$

where $\mathcal{G}_n = \mathcal{F}_{\xi_n}$. Since $\gamma_n \leq \xi_n + 1$, we have

$$P\left\{X(\gamma_n) - X(\xi_n) > 0 \mid \mathcal{G}_n\right\} \leq P\left\{X(\xi_n + 1) - X(\xi_n) > 0 \mid \mathcal{G}_n\right\}$$

$$= 1 - \exp\left\{-\left[\Lambda(\xi_n + 1) - \Lambda(\xi_n)\right]\right\}$$

by virtue of Proposition 1 . But $\Lambda(t+1) - \Lambda(t) = \lambda$ for all $t \geq 0$, and so we have

$$P\{N \geq n+1\} \leq \left(1 - e^{-\lambda}\right) P\{\alpha \geq \xi_n\} = \left(1 - e^{-\lambda}\right) P\{N \geq n\} \quad .$$

Thus, by induction, $P\{N \geq n\} \leq \left(1 - e^{-\lambda}\right)^{n-1}$ for $n \in I$, which implies that $E\{N\} < \infty$. In particular, $\alpha = \gamma_N$ and so we have $\alpha < \infty$ w.p.1 . Note that the total idle time of the server during $[0, \alpha]$ is

$$-\inf\{Y(s-) : 0 \leq s \leq \alpha\} = \sum_{n=1}^{N-1} (\nu_n - \xi_n) + (\gamma_N - \xi_N) \quad .$$

Hence, from the definition of Z , we have

$$0 = Z(\alpha) = Y(\alpha) + \sum_{n=1}^{N-1}(\nu_n - \xi_n) + (\gamma_N - \xi_N) \geq Y(\alpha) \quad .$$

But, the nth idle period has duration $(\nu_n - \xi_n) \leq 1$ for $1 \leq n \leq N - 1$, and $\gamma_N - \xi_N \leq 1$ as well, so that $0 \leq Y(\alpha) + N$. Thus, we have $0 \leq -Y(\alpha) \leq N$ w.p.1, and $P\{-Y(\alpha) > 0\} \geq P\{A(1) = 0\} > 0$, whence $0 < E\{-Y(\alpha)\} \leq E\{N\} < \infty$.

Having shown $P\{\alpha < \infty\} = 1$, we set $\alpha_1 = \alpha$ and then recursively define

$$\alpha_{k+1} = \inf\{n \in I : n > \alpha_k \text{ and } Z(n) = 0\}$$

for $k \in I$. It follows from Proposition 2 that $Y(\alpha_1)$, $Y(\alpha_2) - Y(\alpha_1)$, . . . are i.i.d., as are $\alpha_1, \alpha_2 - \alpha_1, \ldots$. We write

$$(5) \qquad \frac{-Y(\alpha_k)}{k} = \left(\frac{1}{\alpha_k}\sum_{j=1}^{\alpha_k}\Big[Y(j-1) - Y(j)\Big]\right) \cdot \frac{\alpha_k}{k} \quad .$$

Then, letting $k \to \infty$ in (5), we conclude from the strong law of large numbers that $E\{-Y(\alpha)\} = (1 - \rho)E\{\alpha\}$. Thus, $(1 - \rho)E\{\alpha\} \leq E\{N\} < \infty$, which completes the proof.

Remark: If $\rho \geq 1$, one can show $E\{\alpha\} = \infty$ by observing that α is a weak descending ladder index for the random walk $\{Y(n), n \in I\}$.

Throughout the remainder of the paper, we define the integer-valued optional times $\{\alpha_n, n \in I\}$ as in the proof of Proposition 3, and we set $\beta_n = A(\alpha_n)$ for $n \in I$ with $\beta = \beta_1$. Since α is optional and $\{A(n) - A(n-1), n \in I\}$ is an i.i.d. sequence, Wald's Lemma gives $E\{\beta\} = E\{\alpha\} \cdot E\{A(1)\} = \lambda E\{\alpha\} < \infty$. Also, setting $\alpha_0 = \beta_0 \equiv 0$, we define

$$U_n(x) = \int_{\alpha_{n-1}}^{\alpha_n} 1_{\{Z(t) \le x\}}\, dt$$

and

$$V_n(x) = \sum_{k=\beta_{n-1}+1}^{\beta_n} 1_{\{W_n \le x\}}$$

for $n \in I$ and $x \ge 0$, with the convention that an empty sum is zero. (Thus, $V_n(x) = 0$ with positive probability for each fixed n and x.) Let $U(x) = U_1(x)$ and $V(x) = V_1(x)$, and note that $U(x) \le \alpha$ and $V(x) \le \beta$. Since α_n is optional, it follows from Propositions 2 and 3 that $\{Z(\alpha_n + u),\ u \ge 0\}$ is independent of $\mathcal{F}_{\alpha_n}$ and distributed as Z . The following is then immediate.

<u>Proposition</u> 4 . Fix $x \ge 0$. The sequences $\{\alpha_n - \alpha_{n-1},\ n \in I\}$, $\{\beta_n - \beta_{n-1},\ n \in I\}$, $\{U_n(x),\ n \in I\}$ and $\{V_n(x),\ n \in I\}$ are each i.i.d. with finite mean.

3. The Limit Theorems

For each $t \in [0, 1)$ we define a proper d.f. H_t by

$$E\{\alpha\} \cdot H_t(x) = E\left\{\sum_{k=0}^{\alpha-1} 1_{\{Z(k+t) \le x\}}\right\}, \quad x \ge 0 .$$

<u>Proposition</u> 5 . For each $t \in [0, 1)$ and each $x \ge 0$,

$$\lim_{n\to\infty} P\{Z(n+t) \le x\} = H_t(x) .$$

<u>Proof</u>: We have already observed in Section 2 that $\{Z(\alpha + u), u \ge 0\}$ is independent of $\mathscr{F}_\alpha$ and distributed as Z. Thus for $n = 0, 1, 2, \ldots$

$$P\{Z(n+t) \le x\} = P\{\alpha > n, Z(n+t) \le x\} + \sum_{k=1}^{n} P\{\alpha = k\} \cdot P\{Z(n-k+t) \le x\}$$

for fixed $t \in [0, 1)$ and $x \ge 0$. Let $v_n = P\{Z(n+t) \le x\}$, $b_n = P\{\alpha > n, Z(n+t) \le x\}$, and $f_n = P\{\alpha = n\}$ for $n = 0, 1, 2, \ldots$. Then, the equation above can be equivalently expressed as

$$(6) \qquad v_n = b_n + \sum_{k=1}^{n} f_k v_{n-k} , \quad n = 0, 1, 2, \ldots .$$

Now, since $f_1 > 0$ and $E\{\alpha\} < \infty$ it follows from the discrete renewal theorem (cf. Feller (1968), p. 330) that

$$\lim_{n\to\infty} v_n = \sum_{k=0}^{\infty} b_k \Big/ E\{\alpha\}$$

$$= E\left\{\sum_{k=0}^{\infty} 1_{\{\alpha > k\}} 1_{\{Z(k+t) \le x\}}\right\} \Big/ E\{\alpha\}$$

the second equality holding by virtue of monotone convergence. And, the the second expression for the limit is clearly $H_t(x)$, completing the proof.

Proposition 6 . For all $x \geq 0$

$$\frac{1}{t}\int_0^t 1_{\{Z(s) \leq x\}}\, ds \longrightarrow H(x) \qquad \text{w.p.1}$$

as $t \to \infty$ where

$$H(x) = \int_0^1 H_s(x)\, ds \quad .$$

Proof: From Propositions 3 and 4 and the strong law of large numbers it is easy to show that

$$\frac{1}{t}\int_0^t 1_{\{Z(s) \leq x\}}\, ds \longrightarrow E\{U(x)\}/E\{\alpha\} \qquad \text{w.p.1}$$

as $t \to \infty$. Now observe that

$$E\{U(x)\} = E\left\{\sum_{k=0}^{\alpha-1}\int_k^{k+1} 1_{\{Z(s) \leq x\}}\, ds\right\}$$

$$= E\left\{\sum_{k=0}^{\alpha-1}\int_0^1 1_{\{Z(k+s) \leq x\}}\, ds\right\}$$

$$= \int_0^1 E\left\{\sum_{k=0}^{\alpha-1} 1_{\{Z(k+s) \leq x\}}\, ds\right\} \quad ,$$

the last equality following from Fubini's Theorem (cf. Neveu (1965), p. 91). Comparing this with the definition of $H_t(x)$ completes the proof.

Proposition 7 . For all $x \geq 0$

$$\lim_{n \to \infty} P\{W_n \leq x\} = G(x)$$

where

$$G(x) = E\{V(x)\} / E\{\beta\} .$$

Proof: Since α is optional, we see that $\{W_{\beta+n}, n \in I\}$ is independent of $\mathcal{F}_\alpha$ and distributed as W . Thus

$$P\{W_n \leq x\} = P\{\alpha \geq n, W_n \leq x\} + \sum_{k=0}^{n-1} P\{\beta = k\} P\{W_{n-k} \leq x\}$$

for fixed $x \geq 0$ and $n \in I$. Let $v_n = P\{W_n \leq x\}$, $b_n = P\{\beta \geq n, W_n \leq x\}$, and $f_n = P\{\beta = n\}$ for $n = 0, 1, 2, \ldots$, where $v_0 = b_0 = 0$. Then, the equation above has the form (6) with $f_1 > 0$, and again we invoke the discrete renewal theorem to obtain

$$\lim_{n \to \infty} P\{W_n \leq x\} = \sum_{k=1}^{\infty} b_k \Big/ E\{\beta\}$$

$$= E\left\{\sum_{k=1}^{\infty} 1_{\{\beta \geq k\}} 1_{\{W_k \leq x\}}\right\} \Big/ E\{\beta\} .$$

The last expression for the limit clearly equals $E\{V(x)\}/E\{\beta\}$, completing the proof.

Combining Proposition 4 with the strong law of large numbers and the definition of G, we easily obtain the following result.

<u>Proposition 8</u>. For all $x \geq 0$

$$\frac{1}{n}\sum_{k=1}^{n} 1_{\{W_k \leq x\}} \longrightarrow G(x) \qquad \text{w.p.1}$$

as $n \to \infty$.

<u>Proposition 9</u>. $\lambda G(x) = \int_0^1 \lambda(s) H_s(x)\, ds \quad , \quad x \geq 0$.

<u>Proof</u>: Fix $x \geq 0$ and let

$$\gamma(t) = \sum_{k=1}^{A(t)} 1_{\{W_k \leq x\}}$$

for $t > 0$. From Proposition 8 and the fact that $A(t)/t \to \lambda$ w.p.1 as $t \to \infty$, it follows readily that $\gamma(t)/t \to \lambda G(x)$ w.p.1 as $t \to \infty$. Furthermore, $0 \leq \gamma(t)/t \leq A(t)/t$, and $E\{A(t)/t\} = \Lambda(t)/t \leq 2\lambda$ for all $t \geq 1$. Thus, uniform integrability gives $E\{\gamma(t)/t\} \to \lambda G(x)$ as $t \to \infty$. Now let

$$\gamma_n(t) = \sum_{k=0}^{n-1} 1_{\{Z(kt/n) \leq x\}} \left[A((k+1)t/n) - A(kt/n)\right]$$

for $n \in I$ and $t \geq 0$. Then $\gamma_n(t) \to \gamma(t)$ w.p.1 as $n \to \infty$, and hence, by dominated convergence, $E\{\gamma_n(t)\} \to E\{\gamma(t)\}$ as $n \to \infty$. From the independent

increments of X we have

$$E\{\gamma_n(t)\} = \sum_{k=0}^{n-1} P\{Z(kt/n) \leq x\} \cdot \Big[\Lambda((k+1)t/n) - \Lambda(kt/n)\Big]$$

$$\longrightarrow \int_0^t P\{Z(s) \leq x\}\,\lambda(s)\,ds$$

as $n \to \infty$. Thus

$$\begin{aligned}
\lambda G(x) &= \lim_{t\to\infty} \frac{1}{t} E\{\gamma(t)\} \\
&= \lim_{n\to\infty} \frac{1}{n} E\{\gamma(n)\} \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_0^1 P\{Z(k+s) \leq x\}\,\lambda(s)\,ds \\
&= \int_0^1 H_s(x)\,\lambda(s)\,ds \quad ,
\end{aligned}$$

the last equality following from Proposition 5 and bounded convergence.

4. More on the Asymptotic Distributions

We now show how either of the asymptotic distributions H and G can be obtained from the other. We define the d.f. $\hat{F}$ as in Section 1.

Proposition 10 . $H(x) = 1 - \rho + \rho \cdot (G * \hat{F})(x)$, $x \geq 0$.

Proof: In the proof of Proposition 6 we have shown that $H(x) = E\{U(x)\}/E\{\alpha\}$. Let

$$U^*(x) = \int_0^{\alpha} 1_{\{Z(t) > x\}} \, dt \quad , \quad x \geq 0 \quad .$$

Then $U(x) + U^*(x) = \alpha$ for all $x \geq 0$, where $U(x)$ is defined as in Section 2. Furthermore, $U(0)$ and $U^*(0)$ represent, respectively, the total server idle time and the total server busy time during the initial regenerative cycle $[0, \alpha)$. Thus we have

$$(7) \qquad \alpha = U^*(0) + U(0) = \sum_{k=1}^{A(\alpha)} S_k + U(0) \quad ,$$

where S_k is the height of the kth jump of X (the service time of the kth arriving customer) and we follow the convention that an empty sum is zero. Now, from (3) in Lemoine (1974) it then follows that

$$(8) \qquad U(x) = \sum_{k=1}^{A(\alpha)} \min\{S_k, (x - W_k)^+\} + U(0) \quad .$$

Using (7) and (8), we thus have

$$(9) \qquad U^*(x) = \alpha - U(x) = \sum_{k=1}^{A(\alpha)} S_k - \sum_{k=1}^{A(\alpha)} Y_k \quad ,$$

where $Y_k = \min\{S_k , (x - W_k)^+\}$. Using Wald's Lemma we can obtain

$$(10) \qquad E\left\{\sum_{k=1}^{A(\alpha)} S_k\right\} = E\{A(\alpha)\} \cdot E\{S_1\} = \lambda E\{\alpha\} \cdot E\{S_1\} = \rho E\{\alpha\} \quad .$$

Let $\mathscr{H}_0$ denote the σ-algebra in $\mathscr{F}$ generated by T_1, and then for $k \in I$ let $\mathscr{H}_k$ denote the σ-algebra generated by $T_1, S_1, \ldots, T_k, S_k$, and T_{k+1}. Then

$$E\left\{\sum_{k=1}^{A(\alpha)} Y_k\right\} = \sum_{k=1}^{\infty} E\left\{ Y_k \, 1_{\{A(\alpha) > k-1\}}\right\}$$

$$= \sum_{k=1}^{\infty} E\left\{E\left(Y_k \,|\, \mathscr{H}_{k-1}\right) \cdot 1_{\{A(\alpha) > k-1\}}\right\}$$

$$= E\left\{\sum_{k=1}^{A(\alpha)} E\left(Y_k \,|\, \mathscr{H}_{k-1}\right)\right\}$$

$$= E\left\{\sum_{k=1}^{A(\alpha)} E\{S_1\} \cdot \hat{F}[(x - W_k)^+]\right\} .$$

Proceeding exactly as in the argument for (5) in Lemoine (1974), it follows that

$$E\left\{\sum_{k=1}^{A(\alpha)} Y_k\right\} = E\{S_1\} \cdot E\{A(\alpha)\} \cdot (G * \hat{F})(x) = \rho E\{\alpha\} \cdot (G * \hat{F})(x) . \tag{11}$$

Since $E\{U(x)\} = E\{\alpha\} \cdot H(x)$, the desired result follows from (9), (10) and (11). This completes the proof.

The mean of $\hat{F}$ is $E\{S_1^2\}/2E\{S_1\}$, so that Proposition 10 gives

$$\int_0^\infty x\,H(dx) = \lambda E\{S_1^2\}/2 + \rho \int_0^\infty x\,G(dx) \quad .$$

5. Concluding Remarks

In defining our model (Section 2) we have assumed $\Lambda(\cdot)$ to be non-decreasing and absolutely continuous with a periodic density $\lambda(\cdot)$, but the assumption of absolute continuity is not really necessary. We may instead assume we are given a non-decreasing and continuous function Λ with $\Lambda(0) = 0$, $\Lambda(\infty) = \infty$, and $\Lambda(n + t) = n\lambda + \Lambda(t)$ for $n \in I$ and $t \in [0, 1)$. All of our results continue to hold exactly as stated, except that $\lambda(s)ds$ is replaced by $\Lambda(ds)$ in Proposition 9.

This suggests the further generalization where Λ is permitted to have discontinuities. In that case, the customer arrival process A may have jumps of any integer size at time points where Λ is discontinuous. (The size of the jump in A at such a time point is Poisson distributed with mean equal to the height of the jump discontinuity.) Our results for the server load process Z continue to hold, but those for the waiting time process W must be altered somewhat.

With Λ continuous, our model can also be generalized by allowing the service time distribution to depend on the time of day at which a customer arrives. With the traffic intensity ρ properly redefined, all of our results except Proposition 10 continue to hold, but the proof of Proposition 3 becomes much more complicated.

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LIMIT THEOREMS FOR PERIODIC QUEUES

by

J. Michael Harrison
Graduate School of Business
Stanford, University

and

Austin J. Lemoine
Control Analysis Corporation

ABSTRACT

Consider a single server queue with service times distributed as a general random variable S and with nonstationary Poisson input. It is assumed that the arrival rate function $\lambda(\cdot)$ is periodic with average value λ and that $\rho = \lambda E\{S\} < 1$. Both weak and strong limit theorems are proved for the waiting time process $W = \{W_1, W_2, \ldots\}$ and the server load (or virtual waiting time process) $Z = \{Z(t), t \geq 0\}$. The asymptotic distributions associated with Z and W are shown to be related in various ways. In particular, we extend to the case of periodic Poisson input a well known formula (due to Takács) relating the limiting virtual and actual waiting time distributions of a GI/G/1 queue.